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                DISCRETE FOURIER TRANSFORMS FOR SYMMETRIC OR ANTISYMMETRIC REAL--ETC(U)
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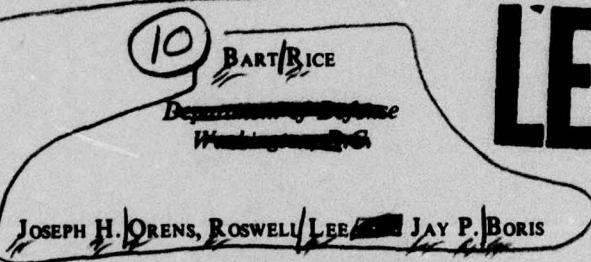
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Discrete Fourier Transforms
for Symmetric or Antisymmetric Real Sequences
With the Point of Symmetry or Antisymmetry
Midway Between Two Cells.

(10) BART RICE

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~~Washington, D.C.~~

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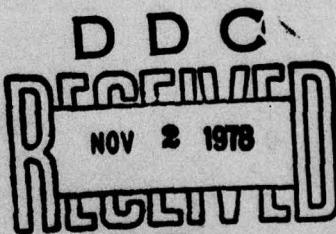
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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

A technique is presented for performing the finite discrete Fourier transforms on real sequences which satisfy either symmetric or antisymmetric boundary conditions and which have the point of symmetry or antisymmetry at the midpoint between two cells. The method approximately halves the number of operations required if symmetry or antisymmetry is ignored.

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DISCRETE FOURIER TRANSFORMS
FOR SYMMETRIC OR ANTISYMMETRIC REAL SEQUENCES
WITH THE POINT OF SYMMETRY OR ANTISYMMETRY
MIDWAY BETWEEN TWO CELLS

1. Introduction

In this paper we present a method for computing the discrete Fourier transform (DFT) of real sequences $\{y(j)\}$ of length $4N$ which satisfy the symmetric or antisymmetric boundary conditions

$$y(4N - j - 1) = sy(j), \qquad 0 \leqslant j \leqslant 4N - 1, s = \pm 1. \tag{1.1}$$

Such sequences appear in several important classes of problems in which certain considerations (physical boundaries, reflection points, etc.) dictate that the symmetry or antisymmetry points be at the midpoint between cells. For example, such sequences are involved in solving Poisson's equation $\nabla^2 \phi = \rho$ for the gravitational potential ϕ_g or the electrostatic potential ϕ_ρ, given the mass or charge density when the source terms are specified half a cell away from the points of symmetry or antisymmetry. The conditions (1.1) reflect this property, supposing symmetry or antisymmetry of the sequence $\{y(j)\}$ at points halfway between $y(-1)$ and $y(0)$, $y(2N-1)$ and $y(2N)$, and $y(4N-1)$ and $y(4N)$. This contrasts with situations for which the usual sine-cosine transforms are applicable, in which $y(0)$, $y(2N)$, and $y(4N)$ are the points of symmetry or antisymmetry and have the symmetric or antisymmetric boundary conditions

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$$y(4N - j) = sy(j), \qquad 0 \leqslant j \leqslant 4N, s = \pm 1. \tag{1.2}$$

The approach presented here is more appropriate for efficient calculation of DFTs of real sequences satisfying (1.1) than the sine-cosine transforms of [1].

Our technique for computing the DFT

$$Y(k) = \sum_{j=0}^{4N-1} y(j)w(jk), \qquad 0 \leqslant k \leqslant 4N - 1, \tag{1.3}$$

where $w(q) = \exp(-2\pi iq/4N)$, approximately halves the number of operations which are required by algorithms in which the added conditions of symmetry or antisymmetry are not taken into account.

2. Derivation of the Algorithm

The DFT of any real sequence $\{y(j)\}$ consisting of an even number of points $2M$ ($M = 2N$ in our case) may be calculated by performing a DFT on M complex points, plus a little overhead. Specifically, if

$$x(j) = y(2j) + iy(2j + 1), \qquad 0 \leqslant j \leqslant M - 1, \tag{2.1}$$

and

$$X(k) = \sum_{j=0}^{M-1} x(j) \exp(-2\pi ijk/M), \qquad 0 \leqslant k \leqslant M - 1, \tag{2.2}$$

then it is a straightforward exercise to verify that

$$Y(k) = [1 - iw(k)]X(k)/2 + [1 + iw(k)]X^*(M - k)/2,$$

$$Y(2M - k) = Y^*(k), \qquad 0 \leqslant k \leqslant M. \tag{2.3}$$

where $X(M) = X(0)$ and p^* denotes the complex conjugate of p. For inversion, the coefficients $\{X(k)\}$ can be constructed from the coefficients $\{Y(k)\}$ and are given by

$$X(k) = [1 + iw(-k)]Y(k)/2 + [1 - iw(-k)]Y^*(M-k)/2, \quad 0 \leq j \leq M-1. \quad (2.4)$$

The boundary values on the sequence $\{x(j)\}$ induced by (1.1) are given by

$$x(M - j - 1) = isx^*(j), \quad 0 \leq j \leq M-1. \quad (2.5)$$

Therefore, for $0 \leq k \leq M-1$,

$$X^*(k) = \sum_{j=0}^{M-1} x^*(j)w(-2jk) = -is \sum_{j=0}^{M-1} x(M-j-1)w(-2jk)$$

$$= -is \sum_{j=0}^{M-1} x(j)w\{-2k(M-j-1)\} = -isw(2k)X(k). \quad (2.6)$$

From (2.5) it follows that (2.2) can be written as a sum of two DFTs, each of length N:

$$X(k) = \sum_{j=0}^{N-1} x(2j)w(4jk) + w(2k) \sum_{j=0}^{N-1} x(2j+1)w(4jk)$$

$$= \sum_{j=0}^{N-1} x(2j)w(4jk) + isw(-2k) \sum_{j=0}^{N-1} x^*(2j)w(-4jk).$$

Therefore, letting

$$B(k) = \sum_{j=0}^{N-1} x(2j)w(4jk), \quad 0 \leq k \leq N-1, \quad (2.7)$$

and noting that $B(N+k) = B(k)$, we have

$$X(k) = B(k) + isw(-2k)B^*(k),$$

$$X(N+k) = B(k) - isw(-2k)B^*(k),$$

$$X(2N-k) = B(N-k) + isw(2k)B^*(N-k), \quad 0 \leq k \leq N-1. \quad (2.8)$$

Hence, all $4N$ Fourier coefficients $\{Y(k)\}$ can be found from a transform of the N complex points $\{x(2j): 0 \leq j \leq N-1\} = \{y(4j) + iy(4j+1): 0 \leq j \leq N-1\}$.

Using the expressions in (2.8) to evaluate the quantities $\{Y(k)\}$ in (2.3), we obtain

$$Y(k) = w(-k/2)[Q(k) + sQ^*(k)]/2, \qquad 0 \leqslant k \leqslant M-1, \tag{2.9}$$

where

$$Q(k) = [w(k/2) - iw(3k/2)]B(k) + [w(k/2) + iw(3k/2)]B^*(N-k),$$

$$Q(N+k) = \{w[(N+k)/2] - iw[3(N+k)/2]\}B(k)$$

$$+ \{w[(N+k)/2] + iw[3(N+k)/2]\}B^*(N-k), \qquad 0 \leqslant k \leqslant N-1. \tag{2.10}$$

We should note that the quantities $Q(k) + sQ^*(k)$ in (2.9) are either real ($s = 1$) or purely imaginary ($s = -1$). Because of the symmetry conditions (1.1) there are only $2N$ linearly independent coefficients. The N complex coefficients $\{B(k)\}$ are one such set, while the $2N$ real or purely imaginary coefficients $\{\tilde{Y}(k)\}$ defined by

$$\tilde{Y}(k) = [Q(k) + sQ^*(k)]/2, \qquad 0 \leqslant k \leqslant M-1, \tag{2.11}$$

are another. Notice that the coefficients $\{Y(k)\}$ and $\{\tilde{Y}(k)\}$ differ only by the phase factor $w(-k/2)$. It is a straightforward matter to show that $\{\tilde{Y}(k)\}$ is the set of Fourier coefficients for the pure sine or cosine expansion of the original sequence, if a coordinate system offset by half a cell is used. This does not allow the coefficients to be readily calculated by the methods of [1], since in this shifted representation the original sequence points are now specified one-half cell from the expansion points. For inversion, the coefficients $\{B(k)\}$ can be expressed in terms of the quantities $\{\tilde{Y}(k)\}$ as follows:

$$4B(k) = [w(-k/2) + iw(-3k/2)]\tilde{Y}(k)$$

$$+ \{w[(N-k)/2] + iw[3(N-k)/2]\}\tilde{Y}^*(N-k)$$

$$+ \{w[-(N+k)/2] + iw[-3(N+k)/2]\}\tilde{Y}(N+k)$$

$$- [iw(-k/2) + w(-3k/2)]\tilde{Y}^*(2N-k), \qquad 0 \leqslant k \leqslant N-1. \tag{2.12}$$

Of course, the decision to utilize a particular one of the sets $\{B(k)\}$, $\{\tilde{Y}(k)\}$, or $\{Y(k)\}$ depends upon the particular problem being treated. If computing speed is the primary consideration,

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then working directly with the quantities $\{B(k)\}$ is the best approach. However, for involved manipulations of the transform coefficients, $\{\tilde{Y}(k)\}$ and $\{Y(k)\}$ are more easily interpreted physically, and use of one of these sets may be more appropriate.

Frequently the DFT is performed to facilitate "smoothing" the data, or "filtering" it in some way. This is often accomplished by multiplying the Fourier coefficients $\{Y(k)\}$ by "form factors" $F(k)$, $0 \leqslant k \leqslant 4N - 1$, which are all real or all purely imaginary, and then performing the inverse DFT

$$z(j) = \frac{1}{4N} \sum_{k=0}^{4N-1} F(k) Y(k) w(-jk), \qquad 0 \leqslant j \leqslant 4N - 1.$$

In the present problem, we impose the restrictions

$$F(k) = F^*(4N - k), \qquad 1 \leqslant k \leqslant 4N - 1. \tag{2.13}$$

We write

$$F^*(k) = tF(k), \qquad 0 \leqslant k \leqslant 4N - 1,$$

where $t = +1$ or -1 according as each form factor is real or purely imaginary, and $F(4N) = F(0)$. Using (2.3) it is easily verified that

$$z^*(j) = z(j), \qquad 0 \leqslant j \leqslant 4N - 1, \tag{2.14}$$

and hence the "smoothed" coefficients $\{z(j)\}$ are real. Using (2.3) and (2.13) it may be directly shown that

$$z(4N - j - 1) = stz(j), \qquad 0 \leqslant j \leqslant 4N - 1. \tag{2.15}$$

Thus, the sequence $\{z(j)\}$ is a member of the symmetry class defined by (1.1), so from (2.5) and (2.7) we may conclude the existence of a sequence $\{B'(k)\}$ such that

$$B'(k) = \frac{1}{N} \sum_{j=0}^{N-1} u(2j) w(4jk), \qquad 0 \leqslant k \leqslant N - 1,$$

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where

$$u(M - j - 1) = istu^*(j),$$

$$u(j) = z(2j) + iz(2j + 1), \qquad 0 \leqslant j \leqslant M - 1.$$

Consequently, the "smoothed" data points $\{z(j)\}$ can be determined from an inverse transform of length N applied to the coefficients $\{B'(k)\}$. Specifically,

$$u(2j) = \frac{1}{N} \sum_{k=0}^{N-1} B'(k) w(-4jk), \qquad 0 \leqslant j \leqslant N - 1.$$

It remains to express the quantities $\{B'(k)\}$ in terms of the coefficients $\{B(k)\}$. From (2.12) we obtain

$$4B'(k) = [w(-k/2) + iw(-3k/2)] \tilde{Y}'(k)$$

$$+ \{w[(N - k)/2] + iw[3(N - k)/2]\} \tilde{Y}'^*$$

$$+ \{w[-(N + k)/2] + iw[-3(N + k)/2]\} \tilde{Y}'(N$$

$$- [iw(-k/2) + w(-3k/2)] \tilde{Y}'^*(2N - k), \qquad 0 \leqslant k \leqslant N - 1, \qquad (2.16)$$

where

$$\tilde{Y}'(k) = F(k) \tilde{Y}(k), \qquad 0 \leqslant k \leqslant M - 1. \qquad (2.17)$$

Substituting (2.17) into (2.16), it is a bit tedious but not difficult to verify that

$$8B'(k) = \alpha_k B(k) + \beta_k B^*(k) + \gamma_k B(N - k) + \delta_k B^*(N - k),$$

where

$$\alpha_k = 2[F(k) + F^*(2N - k) + F(N + k) + F^*(N - k)]$$

$$- i[F(k) - F^*(2N - k)][w(k) - w(-k)]$$

$$- [F(N + k) - F^*(N - k)][w(k) + w(-k)];$$

$$\beta_k = isw(-2k)\{2[F(k) + F^*(2N - k) - F(N + k) - F^*(N - k)]$$

$$-i[F(k) - F^*(2N - k)][w(k) - w(-k)]$$

$$+ [F(N + k) - F^*(N - k)][w(k) + w(-k)]\};$$

$$\gamma_k = isw(-2k)\{-i[F(k) - F^*(2N - k)][w(k) + w(-k)]$$

$$+ [F(N + k) - F^*(N - k)][w(k) - w(-k)]\};$$

$$\delta_k = i[F(k) - F^*(2N - k)][w(k) + w(-k)]$$

$$+ [F(N + k) - F^*(N - k)][w(k) - w(-k)].$$

Consequently, calculation of the quantities $\{\tilde{Y}(k)\}$ or $\{Y(k)\}$ is not necessary. All that is required is to compute the transform $\{B(k)\}$, obtain the quantities $\{B'(k)\}$ using the stored constants $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$, $\{\delta_k\}$, and compute the inverse transform $\{u(2j)\}$.

3. Conclusions

The $4N$ real data points $\{y(j): 0 \leqslant j \leqslant 4N - 1\}$ satisfying the symmetric or antisymmetric boundary conditions (1.1) may be "processed" (transformed, smoothed, and inverse transformed) using two DFTs of length N. This is a savings of approximately a factor of two in the number of arithmetic operations over the computation required by DFT algorithms in which symmetry or antisymmetry is not taken into account. In two and three dimensional applications, we obtain savings factors of four and eight, respectively.

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